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SOLUTION OF NONHOMOGENEOUS PROBLEMS OF HEAT TRANSFER THEORY BY FRACTIONAL DIFFERENTIATION

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The problem of warming up of a semi-infinite region with varying physical parameters and distributed heat sources is considered. A functional expression is derived which makes it possible to determine the temperature gradient at the region boundary in the form of a series in derivatives of fractional order of the boundary temperature. This method was previously proposed [1] for the case of absence of heat sources.

Let us consider the process of heat transfer in a semi-infinite region defined by

$$\left[\frac{\partial}{\partial t} - \alpha(x, t)\frac{\partial^2}{\partial x^2} - \beta(x, t)\frac{\partial}{\partial x} + \gamma(x, t)\right] T = Q(x, t), \quad x \geqslant 0, \quad t > 0 \tag{1}$$

$$T|_{x=0} = T_0(t), \quad T|_{x=\infty} = 0, \quad T|_{t=0} = 0 \tag{2}$$

where $\alpha > 0$, β and $\gamma \geqslant 0$, Q is an analytic function of the x-coordinate and time t, T is the temperature at any arbitrary point, and T_0 is the specified temperature of the boundary. Only the temperature gradient $q^0 = (\partial T / \partial x)_{x=0}$ at the region boundary is to be determined.

Equation (1) may be written as [1]

$$LMT = Q \tag{3}$$

$$L = \sum_{m=0}^{\infty} b_m(x, t)\frac{\partial^{(1-m)/2}}{\partial t^{(1-m)/2}} - \alpha^{1/2}\frac{\partial}{\partial x}$$

$$M = \sum_{n=0}^{\infty} a_n(x, t)\frac{\partial^{(1-n)/2}}{\partial t^{(1-n)/2}} + \alpha^{1/2}\frac{\partial}{\partial x}$$

where a_n and b_m are known functions defined in [1] in terms of α, β, γ and their derivatives. The operators of fractional differentiation of order ν are defined as in [2] by

$$\frac{d^\nu f(t)}{dt^\nu} = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{-\nu}d\tau \tag{4}$$

where $(f(t)$ is an arbitrary function. The basic properties of operation (4) are

$$\frac{d^\mu}{dt^\mu}\frac{d^\nu}{dt^\nu} f(t) = \frac{d^{\mu+\nu}}{dt^{\mu+\nu}} f(t), \quad \mu + \nu \leqslant 1 \tag{5}$$

$$\frac{d^\nu}{dt^\nu} f(t) g(t) = \sum_{n=0}^{\infty} \binom{\nu}{n} f^{(n)}(t)\frac{d^{\nu-n} g(t)}{dt^{\nu-n}}$$

Let us now consider instead of (3) the equation

$$MT = Q^* \qquad (LQ^* = Q) \tag{6}$$

The auxilliary function Q^* can be defined by

$$Q^* = LL^{-1}Q^* = L^{-1}Q \tag{7}$$

where L^{-1} is an inverse operator of L for multiplying by the latter on the right.

Solutions of Eq. (6) are also solutions of the input equation (1), as can be verified by multiplying (6) on the left by operator L. It can be shown that the solution of Eq. (6) also satisfies conditions (2), for example, in the case when

$$\lim x \to \infty, \ \alpha, \ \beta, \ \gamma = \text{const}, \qquad \lim x \to \infty, \quad Q = 0, \quad T_0(0) = 0$$

If the series are absolutely and uniformly convergent with respect to x when $x \to +0$, then by writing (6) for $x = 0$, we directly obtain the solution of the stated problem, i.e. the expression for the temperature gradient at the boundary in terms of the boundary temperature

$$-\alpha^{1/2}(0, t)\frac{\partial T}{\partial x}\bigg|_{x=0} = \sum_{n=0}^{\infty} a_n(0, t)\frac{d^{(1-n)/2}T_0(t)}{dt^{(1-n)/2}} - Q^*(0, t) \tag{8}$$

Let us explain how Q^* is determined from (7). We seek operator L^{-1} of the form

$$L^{-1} = \sum_{s=0}^{\infty} G_s\left(x, t, \frac{\partial^r}{\partial x^r}\right)\frac{\partial^{-(1+s)/2}}{\partial t^{-(1+s)/2}}, \qquad r \leqslant s \tag{9}$$

where G_s are so far unknown operators. Substituting (9) into (7) and using formulas (5), we multiply the operators by each other and obtain a symbolic equation for the determination of operators G_s

$$\Bigg[\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\frac{1+s}{2}}{k} b_m \frac{\partial^k G_s}{\partial t^k}\frac{\partial^{-(m+s)/2-k}}{\partial t^{-(m+s)/2-k}} - \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\frac{1+s}{2}}{k}\alpha^{1/2}\frac{\partial}{\partial x}\frac{\partial^k G_s}{\partial t^k}\frac{\partial^{-(1+s)/2-k}}{\partial t^{-(1+s)/2-k}}\Bigg] Q = Q$$

Equating terms of identical exponents of the derivatives with respect to time, we obtain a symbolic system of recurrent relations for the determination of operators G_s in terms of the known functions b_m and α

$$b_0G_0Q = Q \tag{10}$$

$$\left(b_0G_1 + b_1G_0 - \alpha^{1/2}\frac{\partial G_0}{\partial x}\right)\frac{\partial^{-1/2}Q}{\partial t^{-1/2}} = 0$$

$$\left(b_0G_2 + b_1G_1 + b_2G_0 - \frac{1}{2}b_0\frac{\partial G_0}{\partial t} - \alpha^{1/2}\frac{\partial G_1}{\partial x} + \frac{1}{2}\alpha^{1/2}\frac{\partial}{\partial x}\frac{\partial G_0}{\partial t}\right)\frac{\partial^{-1}Q}{\partial t^{-1}} = 0$$

. . . .

$$\Bigg[\sum_{m=0}^{p-m-s\geqslant 0}\sum_{s=0}\binom{-\frac{1+s}{2}}{\frac{p-m-s}{2}} b_m \frac{\partial^{(p-m-s)/2}G_s}{\partial t^{(p-m-s)/2}} - \sum_{s=0}^{p-1-s\geqslant 0}\binom{-\frac{1+s}{2}}{\frac{p-1-s}{2}}\alpha^{1/2}\frac{\partial}{\partial x}\frac{\partial^{(p-1-s)/2}}{\partial t^{(p-1-s)/2}}\Bigg]\frac{\partial^{-p/2}Q}{\partial t^{-p/2}} = 0$$

. . . .

Using these we obtain explicit expressions for operators G_s

$$G_0 = \frac{1}{b_0}, \quad G_1 = \left(\alpha^{1/2}\frac{\partial G_0}{\partial x} - b_1 G_0\right)$$
$$G_2 = \left(\frac{1}{2} b_0 \frac{\partial G_0}{\partial t} + \alpha^{1/2}\frac{\partial G_1}{\partial x} - \frac{1}{2}\alpha^{1/2}\frac{\partial}{\partial x}\frac{\partial G_0}{\partial t} - b_1 G_1 - b_2 G_0\right) \Big/ b_0$$

. . . .

Example. Let $\alpha = (a + bx)^{4/3}$, $\beta = {}^4/_3\, b\,(a + bx)^{1/3}$ and $\gamma = 0$. Then (3) assumes the form (see [1])

$$\left[\frac{\partial^{1/2}}{\partial t^{1/2}} - \frac{b}{3}(a+bx)^{-1/3} - (a+bx)^{2/3}\frac{\partial}{\partial x}\right] \times$$
$$\left[\frac{\partial^{1/2}}{\partial t^{1/2}} + \frac{b}{3}(a+bx)^{-1/3} + (a+bx)^{2/3}\frac{\partial}{\partial x}\right] T = Q$$

Using (10) we obtain for (9)

$$L^{-1} = \sum_{s=0}^{\infty}\left[\frac{b}{3}(a+bx)^{-1/3} + (a+bx)^{2/3}\right]^s \frac{\partial^{-(1+s)/2}}{\partial t^{-(1+s)/2}}$$

Solution (8) is defined by the expression

$$-a^{2/3} q_0(t) = \frac{d^{1/2} T_0(t)}{dt^{1/2}} + \frac{b T_0(t)}{3a^{1/3}} -$$
$$\sum_{s=0}^{\infty}\left[\frac{b}{3}(a+bx)^{-1/3} + (a+bx)^{2/3}\frac{\partial}{\partial x}\right]^s \frac{\partial^{-(1+s)/2} Q(x,t)}{\partial t^{-(1+s)/2}}\Bigg|_{x=0}$$

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